The neutrosophic number generalized weighted power averaging operator and its application in multiple attribute group decision making

Peide Liu [a,b,*], Xi Liu [a,]

[a,]*School of Management Science and Engineering, Shandong University of Finance and Economics, Jinan Shandong 250014, China*

[b]*School of Economics and Management, Civil Aviation University of China, Tianjin 300300, China*

**The corresponding author: peide.liu@gmail.com*

Abstract: Neutrosophic number (NN) is a useful tool which is used to overcome the difficulty of describing indeterminate evaluation information. The purpose of the study is to propose some power aggregation operators based on neutrosophic number which is used to deal with multiple attributes group decision making problems more effectively. Firstly, the basic concepts and the operational rules and the characteristics of NNs are introduced. Then, some aggregation operators based on neutrosophic numbers are developed, included the neutrosophic number weighted power averaging (NNWPA) operator, the neutrosophic number weighted geometric power averaging (NNWGPA) operator, the generalized neutrosophic number weighted power averaging (GNNWPA)operator. At the same time, the properties of above operators are studied such as idempotency, monotonicity, boundedness and so on. Then, the generalized neutrosophic number weighted power averaging (GNNWPA) operator is applied to solve multiple attribute group decision making problems. Afterwards, a numerical example is given to demonstrate the effective of the new developed method, and some comparison are conducted to verify the influence of different parameters or to reveal the difference with another method. In the end, the main conclusion of this paper is summarized.

Keywords: multiple attribute group decision making; neutrosophic numbers; power aggregation operator; neutrosophic numbers power aggregation operator.

1. Introduction

In real decision making, since the fuzziness and complexity of decision making problems, sometimes the people's judgments by crisp numbers have difficulty in conveying their opinions thoroughly. Zadeh [1] innovatively proposed the fuzzy set (FS) to cope with the fuzzy information. Since the fuzzy set has only the membership degree and has not the non-membership degree, Atanassov [2] made an improvement to overcome this shortcoming, and proposed the intuitionistic fuzzy set (IFS) which is made up with membership degree and non-membership degree. However IFS did not consider the indeterminacy-membership degree. To find a more precise measurement, Smarandache [3] further proposed the neutrosophic numbers (NNs), and it can be divided into determinate part and indeterminate part. The neutrosophic number (NN) is in the form of $N = a + bI$. As we can see that a is the determinate part and bI represents the indeterminate part. Obviously, about the indeterminate part, the fewer it is, the better it is. So, the worst scenario is $N = bI$. Conversely, the best case is $N = a$. To this day, there is the little progress to cope with indeterminate problems by neutrosophic numbers in fields of scientific and engineering techniques. Therefore, it is necessary to propose a new method based on neutrosophic numbers (NNs) to handle group decision making problems.

Researchers have paid more and more attentions on information aggregation operators. The OWA operator can weight the inputs according to the ranking position of them, then many extensions of the OWA operator have been proposed, such as uncertain aggregation operators [4-6], the induced

aggregation operators [7,8], the linguistic aggregation operators [9-11], the uncertain linguistic aggregation operators [12,13,14], the fuzzy aggregation operators [15,16], the fuzzy linguistic aggregation operators [17], the induced linguistic aggregation operators [18], the induced uncertain linguistic aggregation operators [19,20], the fuzzy induced aggregation operators [21] and the intuitionistic fuzzy aggregation operators [22]. Based on the operators mentioned above, Xu and Chen [23] proposed some interval-valued intuitionistic fuzzy arithmetic aggregation (IVIFAA) operators, such as the interval-valued intuitionistic fuzzy weighted aggregation(IVIFWA) operator, the interval-valued intuitionistic fuzzy ordered weighted aggregation (IVIFOWA) operator, and the interval-valued intuitionistic fuzzy hybrid aggregation (IVIFHA) operator. Zhao [24] proposed the generalized intuitionistic fuzzy weighted aggregation (GIFWA) operator [25], the generalized intuitionistic fuzzy ordered weighted (GIFOWA) aggregation operator, and the generalized intuitionistic fuzzy hybrid aggregation (GIFHA) operator. However, these operators didn't consider the relationship between the attributes. So, Yager [26] developed a power average (PA) operator to overcome this shortcoming, i.e., it can consider the relationship between the attributes, a large amount of operators based on PA have been developed to aggregate evaluation information in order to adapt to various environments [15, 27-31]. For instance, power geometric (PG) operator, generalized power average (GPA) operator [30], linguistic generalized power average (LGPA) operator [31] and so on.

To this day, there is not the research on the combination the neutrosophic numbers with power aggregation operator. Thus, it is very necessary to do the research based on neutrosophic numbers aggregation operators. In this study, we will propose the generalized hybrid weighted power averaging operator under neutrosophic numbers environment, and then propose a new method for the multiple attribute group decision problems, which has two advantages, one is that it can cope with the indeterminacy of evaluation information precisely; another is that it can take the relationship between the attributes into consideration.

This paper is written as below: The section 2 is about basic concepts, the operational rules and the characteristics of NNs. In section 3, some aggregation operators based on neutrosophic numbers are developed, such as the neutrosophic number weighted power averaging (NNWPA) operator, the neutrosophic number weighted geometric power averaging (NNWGPA) operator, the generalized neutrosophic number weighted power averaging (GNNWPA) operator, and then their properties are proved. In section 4, we propose a multiple attribute group decision making method based on the GNNWPA operator, and introduce the decision steps. In section 5, a numerical example is given to demonstrate the effective of the new developed method. In section 6, the conclusion is made.

2. Preliminaries

2.1 Basic concepts of neutrosophic numbers and their operators

The concept of neutrosophic number is firstly proposed by Smarandache in neutrosophic probability. It includes two parts: determinate part and indeterminate part.

Definition 1[32-34]. Let $I \in [\beta^-, \beta^+]$ be an indeterminate part, a neutrosophic number N is denoted as:

$$N = a + bI \tag{1}$$

where a and b are both real numbers, and I is the indeterminate part, such that $I^2 = I$, $0 \cdot I = 0$, and I/I = undefined.

Definition 2[32-34]. Let $N_1 = a_1 + b_1 I$ and $N_2 = a_2 + b_2 I$ be two neutrosophic numbers, then, operational

relations of neutrosophic numbers are shown as follows:

(1) $N_1 + N_2 = a_1 + a_2 + (b_1 + b_2)I$ (2)

(2) $N_1 - N_2 = a_1 - a_2 + (b_1 - b_2)I$ (3)

(3) $N_1 \times N_2 = a_1 a_2 + (a_1 b_2 + b_1 a_2 + b_1 b_2)I$ (4)

(4) $N_1{}^2 = (a_1 + b_1 I)^2 = a_1{}^2 + (2a_1 b_1 + b_1{}^2)I$ (5)

(5) $\lambda N_1 = \lambda a_1 + \lambda b_1 I$ (6)

(6) $N_1^{\lambda} = a_1^{\lambda} + \left((a_1 + b_1)^{\lambda} - a_1^{\lambda}\right)I \quad \lambda > 0$ (7)

(7) $\dfrac{N_1}{N_2} = \dfrac{a_1 + b_1 I}{a_2 + b_2 I} = \dfrac{a_1}{a_2} + \dfrac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)} I \quad for\ a_2 \neq 0\ and\ a_2 \neq -b_2$ (8)

Theorem 1. Let $N_i = a_i + b_i I$ be any neutrosophic number, $\lambda, \lambda_1, \lambda_2 > 0$, the operational laws have the following characteristics:

(1) $N_1 \oplus N_2 = N_2 \oplus N_1$ (9)

(2) $N_1 \otimes N_2 = N_2 \otimes N_1$ (10)

(3) $\lambda(N_1 \oplus N_2) = \lambda N_1 \oplus \lambda N_2$ (11)

(4) $\lambda_1 N_1 \oplus \lambda_2 N_1 = (\lambda_1 \oplus \lambda_2) N_1$ (12)

(5) $N_1^{\lambda} \otimes N_2^{\lambda} = (N_1 \otimes N_2)^{\lambda}$ (13)

(6) $N_1^{\lambda_1} \otimes N_1^{\lambda_2} = (N_1)^{\lambda_1 + \lambda_2}$ (14)

Proof.

(1) Obviously, the equation (9) is right according to the operational rule (1) expressed by (2).

(2) Obviously, the equation (10) is right according to the operational rule (3) expressed by (4).

(3) For the left of the equation (11), we have

$$\lambda(N_1 \oplus N_2) = \lambda\left((a_1 + b_1 I) \oplus (a_2 + b_2 I)\right) = \lambda\left((a_1 + a_2) + (b_1 + b_2)I\right)$$

And for the right of the equation (11), we have

$$\begin{aligned}\lambda N_1 \oplus \lambda N_2 &= \lambda(a_1 + b_1 I) \oplus \lambda(a_2 + b_2 I) = (\lambda a_1 + \lambda b_1 I) \oplus (\lambda a_2 + \lambda b_2 I)\\ &= (\lambda a_1 + \lambda a_2) + (\lambda b_1 + \lambda b_2)I = \lambda\left((a_1 + a_2) + (b_1 + b_2)I\right)\end{aligned}$$

So, we can get equation (11) is right.

(4) For the equation (12), we have

$$\begin{aligned}\lambda_1 N_1 \oplus \lambda_2 N_1 &= \lambda_1(a_1 + b_1 I) + \lambda_2(a_1 + b_1 I) = (\lambda_1 a_1 + \lambda_2 a_1) + (\lambda_1 b_1 + \lambda_2 b_1)I\\ &= (\lambda_1 + \lambda_2)a_1 + (\lambda_1 + \lambda_2)b_1 I = (\lambda_1 + \lambda_2)N_1\end{aligned}$$

So, the equation (12) is right.

(5) For the left of the equation (13), we have

$$N_1{}^{\lambda} \otimes N_2{}^{\lambda} = \left(a_1{}^{\lambda} + \left((a_1 + b_1)^{\lambda} - a_1{}^{\lambda}\right)I\right) \otimes \left(a_2{}^{\lambda} + \left((a_2 + b_2)^{\lambda} - a_2{}^{\lambda}\right)I\right)$$

$$= a_1{}^{\lambda} a_2{}^{\lambda} + a_1{}^{\lambda}\left((a_2 + b_2)^{\lambda} - a_2{}^{\lambda}\right)I + a_2{}^{\lambda}\left((a_1 + b_1)^{\lambda} - a_1{}^{\lambda}\right)I + \left((a_2 + b_2)^{\lambda} - a_2{}^{\lambda}\right)\left((a_1 + b_1)^{\lambda} - a_1{}^{\lambda}\right)I$$

$$= a_1{}^{\lambda} a_2{}^{\lambda} + \left(a_1{}^{\lambda}(a_2 + b_2)^{\lambda} - a_1{}^{\lambda} a_2{}^{\lambda}\right)I + \left(a_2{}^{\lambda}(a_1 + b_1)^{\lambda} - a_2{}^{\lambda} a_1{}^{\lambda}\right)I$$

$$+ \left((a_2 + b_2)^{\lambda}(a_1 + b_1)^{\lambda} - a_2{}^{\lambda}(a_1 + b_1)^{\lambda} - a_1{}^{\lambda}(a_2 + b_2)^{\lambda} + a_1{}^{\lambda} a_2{}^{\lambda}\right)I$$

$$= (a_1 a_2)^{\lambda} + \left((a_2 + b_2)^{\lambda}(a_1 + b_1)^{\lambda} - a_1{}^{\lambda} a_2{}^{\lambda}\right)I$$

and the right of the equation (13), we have

$$(N_1 \otimes N_2)^{\lambda} = \left((a_1+b_1)I \otimes (a_2+b_2)I\right)^{\lambda} = \left(a_1a_2 + (a_1b_2 + a_2b_1 + b_1b_2)I\right)^{\lambda}$$

$$= (a_1a_2)^{\lambda} + \left((a_1a_2 + a_1b_2 + a_2b_1 + b_1b_2)^{\lambda} - (a_1a_2)^{\lambda}\right)I$$

$$= (a_1a_2)^{\lambda} + \left((a_1+b_1)^{\lambda}(a_2+b_2)^{\lambda} - a_1^{\lambda}a_2^{\lambda}\right)I$$

So, the equation (13) is right.

(6) For the equation (12), we have

$$N_1^{\lambda_1} \otimes N_1^{\lambda_2} = \left(a_1^{\lambda_1} + \left((a_1+b_1)^{\lambda_1} - a_1^{\lambda_1}\right)I\right) \otimes \left(a_1^{\lambda_2} + \left((a_1+b_1)^{\lambda_2} - a_1^{\lambda_2}\right)I\right)$$

$$= a_1^{\lambda_1}a_1^{\lambda_2} + \left(a_1^{\lambda_1}\left((a_1+b_1)^{\lambda_2} - a_1^{\lambda_2}\right)I + a_1^{\lambda_2}\left((a_1+b_1)^{\lambda_1} - a_1^{\lambda_1}\right)I + \left((a_1+b_1)^{\lambda_2} - a_1^{\lambda_2}\right)\left((a_1+b_1)^{\lambda_1} - a_1^{\lambda_1}\right)I\right)$$

$$= a_1^{\lambda_1}a_1^{\lambda_2} + \left((a_1+b_1)^{\lambda_2}(a_1+b_1)^{\lambda_1} - a_1^{\lambda_2}a_1^{\lambda_1}\right)I$$

$$= a_1^{\lambda_1+\lambda_2} + \left((a_1+b_1)^{\lambda_1+\lambda_2} - a_1^{\lambda_1+\lambda_2}\right)I$$

$$= N_1^{\lambda_1+\lambda_2}$$

So we can get the equation (14) is right.

Definition 3[35]. Suppose that $N_i = a_i + b_i \cdot I$ with $I \in [\beta^-, \beta^+] (i = 1,2,\ldots,n)$ is any neutrosophic number for $a_i, b_i, \beta^-, \beta^+ \in R$, where R is the set of real numbers. To normalize N_i, we get

$$N_i = \frac{a_i}{\max(a_i)} + \frac{b_i}{\max(b_i)} I \tag{15}$$

Definition 4[35]. Suppose that $N_i = a_i + b_i \cdot I$ with $I \in [\beta^-, \beta^+] (i = 1,2,\ldots,n)$ is any neutrosophic number for $a_i, b_i, \beta^-, \beta^+ \in R$, where R is the set of real numbers. We can give the distance between N_i and N_j as follow:

$$d(N_i, N_j) = \frac{1}{2}\sqrt{\frac{[(a_j - a_i) + (b_j - b_i)\beta^-]^2 + [(a_j - a_i) + (b_j - b_i)\beta^+]^2}{2}} \tag{16}$$

which meets the following criteria:

(1) $0 \le d(N_i, N_j) \le 1$ (17)

(2) $d(N_i, N_i) = 0$ (18)

(3) $d(N_1, N_2) = d(N_2, N_1)$ (19)

(4) $d(N_1, N_2) + d(N_2, N_3) \ge d(N_1, N_3)$ (20)

Definition 5[36]. Let $N_i = a_i + b_i I$ be a set of neutrosophic number, $I \in [\beta^-, \beta^+] (i = 1,2,\ldots,n)$, a_i, b_i, $\beta^-, \beta^+ \in R$, where R is the set of real numbers, the neutrosophic number $N_i \in [a_i + b_i\beta^-, a_i + b_i\beta^+]$,

so the possibility degree is

$$P_{ij}=P(N_i\geq N_j)=\max\left\{1-\max\left(\frac{(a_j+b_j\beta^+)-(a_i+b_i\beta^-)}{(a_i+b_i\beta^+)-(a_i+b_i\beta^-)+(a_j+b_j\beta^+)-(a_j+b_j\beta^-)},0\right),0\right\} \quad (21)$$

Where, $P_{ij}\geq 0$, $P_{ij}+P_{ji}=1$, and $P_{ii}=0.5$. Then, the value of N_i $(i=1,2,...,n)$ can be used for ranking order as follows:

$$q_i=\frac{\left(\sum_{j=1}^{n}P_{ij}+\frac{n}{2}-1\right)}{n(n-1)} \quad (22)$$

Therefore, if the value of $q_i\,(i=1,2,\ldots,n)$ is bigger, information that neutrosophic numbers represent is more precise. In consequence, we rank the neutrosophic numbers of $q_i\,(i=1,2,\ldots,n)$ in an ascending order in order to get the best N_i $(i=1,2,...,n)$.

2.2 The Power Aggregation (PA) operator

Definition 5[6]. For real numbers $a_i\,(i=1,2,\ldots,n)$, the power average operator is defined as

$$PA(a_1,a_2,\cdots,a_n)=\frac{\sum_{i=1}^{n}\left(1+T(a_i)\right)\cdot a_i}{\sum_{i=1}^{n}\left(1+T(a_i)\right)} \quad (23)$$

where

$$T(a_i)=\sum_{\substack{j=1\\ j\neq i}}^{n}\sup(a_i,a_j) \quad (24)$$

and $\sup(a_i,a_j)$ means the degree to which a_j supports a_i. It satisfies the following rules.

(1) $\sup(a_i,a_j)=\sup(a_j,a_i)$ (25)

(2) $\sup(a_i,a_j)\in[0,1]$ (26)

(3) $\sup(a_i,a_j)\geq\sup(a_m,a_n)$, if $|a_i - a_j|\leq|a_m - a_n|$ (27)

3. Neutrosophic Number Aggregation Operators

A neutrosophic number includes two parts: determinate part and indeterminate part. Thus, it is a good tool to express the indeterminate and incomplete information. At the same time, the Power aggregation can take the relationship between the attributes into consideration. For this reason, we combine them together, and develop some kinds of neutrosophic number aggregation operators.

3.1 The Neutrosophic Number Weighted Power Averaging Operator

Definition 6[6]. Let $N_i=a_i+b_iI$ be a set of neutrosophic numbers, then we define NNPA (neutrosophic number powered aggregation) operator as follows:

$$NNPA(N_1,\ N_2,\ldots N_n)=\frac{\sum_{i=1}^{n}\left(1+T(N_i)\right)\cdot N_i}{\sum_{i=1}^{n}\left(1+T(N_i)\right)} \tag{28}$$

where $T\left(N_i\right)=\sum_{\substack{j=1\\ j\neq i}}^{n}\sup\left(N_i,N_j\right)$, and $\sup(N_i,N_j)$ means the support for a_i from a_j , $\sup(N_i,N_j)=1-d(N_i,N_j)$. Obviously, it satisfies the following rules:

(1) $\sup(N_i,N_j)=\sup(N_j,N_i)$ (29)

(2) $\sup(N_i,N_j)\in[0,1]$ (30)

(3) $\sup(N_i,N_j)\geq\sup(N_m,N_n)$, if $\left|N_i-N_j\right|\leq\left|N_m-N_n\right|$ (31)

Theorem 2. Let $N_i=a_i+b_iI$ be a set of neutrosophic numbers and NNPA: $NNS^n\rightarrow NNS$. If

$$NNPA(N_1,\ N_2,\ldots N_n)=\frac{\sum_{i=1}^{n}\left(1+T(N_i)\right)\cdot N_i}{\sum_{i=1}^{n}\left(1+T(N_i)\right)}=\frac{\sum_{i=1}^{n}\left(1+T(N_i)\right)\cdot a_i}{\sum_{i=1}^{n}\left(1+T(N_i)\right)}+\frac{\sum_{i=1}^{n}\left(1+T(N_i)\right)\cdot b_i}{\sum_{i=1}^{n}\left(1+T(N_i)\right)}I \tag{32}$$

So the result of Eq.(28) is still a NN.

We use Mathematical induction on n to testify the Eq.(32) as follows:

Proof.

(i) When $n=1$, it's clear that the Eq. (32) is right.

(ii) Suppose when $n=k$, the Eq.(32) is right, i.e.,

$$NNPA(N_1,\ N_2,\ldots N_n)=\frac{\sum_{i=1}^{n}\left(1+T(N_i)\right)\cdot N_i}{\sum_{i=1}^{n}\left(1+T(N_i)\right)}=\frac{\sum_{i=1}^{n}\left(1+T(N_i)\right)\cdot a_i}{\sum_{i=1}^{n}\left(1+T(N_i)\right)}+\frac{\sum_{i=1}^{n}\left(1+T(N_i)\right)\cdot b_i}{\sum_{i=1}^{n}\left(1+T(N_i)\right)}I$$

Then when $n=k+1$, we have

$$NNPA(N_1,N_2,\ldots N_{k+1})=\frac{\sum_{i=1}^{k}\left(1+T(N_i)\right)\cdot a_i}{\sum_{i=1}^{k}\left(1+T(N_i)\right)}+\frac{\sum_{i=1}^{k}\left(1+T(N_i)\right)\cdot b_i}{\sum_{i=1}^{k}\left(1+T(N_i)\right)}I$$

$$+\frac{\left(1+T(N_{k+1})\right)\cdot a_{k+1}}{\sum_{i=1}^{k+1}\left(1+T(N_i)\right)}+\frac{\left(1+T(N_{k+1})\right)\cdot b_{i+1}I}{\sum_{i=1}^{k+1}\left(1+T(N_i)\right)}=\frac{\sum_{i=1}^{k+1}\left(1+T(N_i)\right)\cdot N_i}{\sum_{i=1}^{k+1}\left(1+T(N_i)\right)}$$

Thus, when $n=k+1$, the Eq. (32) is right too.

Accordingly, we can get that the Eq.(32) is right for all n.

Theorem 3. If $Sup\left(\tilde{a}_k, \tilde{a}_j\right) = c$, then the NNPA operator will be reduced to the arithmetic averaging operator of NNs shown as follows.

$$NNPA\left(N_1, N_2, ..., N_n\right) = \frac{\sum_{i=1}^{n} N_i}{n} = \frac{1}{n}\sum_{i=1}^{n} a_i + \frac{1}{n}\sum_{i=1}^{n} b_i I$$

In the following, we can prove the NNPA operator has some desirable characteristics, such as idempotency, monotonicity, boundedness and commutativity.

Theorem 4. (Idempotency).

Let all $N_i = a + b \cdot I$, $i = (1, 2, \ldots, n)$, then

$$NNPA(N_1, N_2, \ldots N_n) = N_i = a + bI$$

Proof.

Since all $N_i = a + b \cdot I$, we have

$$NNPA(N_1,\ N_2, \ldots N_n) = \frac{\sum_{i=1}^{n}\left(1+T(N_i)\right)\cdot N_i}{\sum_{i=1}^{n}\left(1+T(N_i)\right)} = \frac{\sum_{i=1}^{n}\left(1+T(N_i)\right)\cdot a}{\sum_{i=1}^{n}\left(1+T(N_i)\right)} + \frac{\sum_{i=1}^{n}\left(1+T(N_i)\right)\cdot b}{\sum_{i=1}^{n}\left(1+T(N_i)\right)} I = a + bI$$

which completes the proof of this theorem 4.

Theorem 5. (Monotonicity).

Let $N_i = a_i + b_i$ and $N_i^* = a_i^* + b_i^*$ be two collections of NNs which meets $a_i \le a_i^*$ $b_i^* \le b_i$, $i = 1, 2, \ldots, n$, then

$$NNPA(N_1, N_2, \ldots N_n) \le NNPA(N_1^*, N_2^*, \ldots N_n^*).$$

Proof.

Since for all i, $a_i \le a_i^*$, $b_i^* \le b_i$, we can obtain

$$\sum_{i=1}^{n} a_i \le \sum_{i=1}^{n} a_j^*,\quad \sum_{i=1}^{n} b_i^* I \le \sum_{i=1}^{n} b_i\ I$$

So, we can get

$$NNPA(N_1, N_2, \ldots N_n) \le NNPA(N_1^*, N_2^*, \ldots N_n^*)$$

which completes the proof of theorem 5.

Theorem 6. (Boundedness).

Let $N_i = a_i + b_i I\ \ (i = 1, 2, ..., n)$ be a set of NNs. If

$$N_{\max} = \max(N_1, N_2, ..., N_n) = a_{\max} + b_{\min} I,$$
$$N_{\min} = \min(N_1, N_2, ..., N_n) = a_{\min} + b_{\max} I,$$

then

$$N_{\min} \le NNPA(N_1, N_2, \ldots N_n) \le N_{\max}.$$

Proof.

Since $a_{\min} \le a_j \le a_{\max}$, $b_{\min} \le b_j \le b_{\max}$, in the case of all i ,we can obtain

$$\sum_{i=1}^{n} a_{\min} \le \sum_{i=1}^{n} a_j \le \sum_{i=1}^{n} a_{\max} , \sum_{i=1}^{n} b_{\min} \le \sum_{i=1}^{n} b_j \le \sum_{i=1}^{n} b_{\max}$$

So, we can get

$$NNPA(N_{\min}, N_{\min}, \ldots N_{\min}) \le NNPA(N_1, N_2, \ldots N_n) \le NNPA(N_{\max}, N_{\max}, \ldots N_{\max})$$

Based on theorem 3, we can know

$$NNPA(N_{\min}, N_{\min}, \ldots N_{\min}) = N_{\min}$$

$$NNPA(N_{\max}, N_{\max}, \ldots N_{\max}) = N_{\max}$$

So, we can get

$$N_{\min} \le NNPA(N_1, N_2, \ldots N_n) \le N_{\max} .$$

Theorem 7. (Commutativity).

We assume that $(N'_1, N'_2, \ldots, N'_n)$ is any permutation of $(N_1, N_2, \ldots, N_n)$,

then

$$NNPA(N'_1, N'_2, \ldots N'_n) = NNPA(N_1, N_2, \ldots N_n)$$

Proof.

Since $(N'_1, N'_2, \ldots, N'_n)$ is any permutation of $(N_1, N_2, \ldots, N_n)$, we have

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} a'_j , \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} b'_j$$

then, we can get

$$NNPA(N'_1, N'_2, \ldots N'_n) = NNPA(N_1, N_2, \ldots N_n)$$

So, theorem 7 is right.

Definition 7[37]. Let $N_i = a_i + b_i I$ be a set of neutrosophic numbers, and NNWPA:NNS^n→ NNS. If

$$NNWPA(N_1, N_2, \ldots N_n) = \frac{\sum_{i=1}^{n} \omega_i (1 + T(N_i)) N_i}{\sum_{i=1}^{n} \omega_i (1 + T(N_i))} \tag{33}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weight vector of $N_i (i = 1, 2, \ldots n)$ which satisfies $\omega_i \in [0,1]$ $(i = 1, 2, \ldots n)$ and $\sum_{i=1}^{n} \omega_i = 1$. NNWPA operator is called neutrosophic number weighted power averaging operator.

Theorem 8. Let $N_i = a_i + b_i I \ (i = 1,2,..., n)$ be a set of NNs, $\omega = (\omega_1, \omega_2, . . . , \omega_n)$ be the weight vector of $N_i (i = 1,2,\ldots n)$ satisfying $\omega_i \in [0,1] \ (i = 1,2,\ldots,n)$ and $\sum_{i=1}^{n} \omega_i = 1$.Then the result aggregated from Definition 7 is still a NN, even

$$NNWPA(N_1, \ N_2,\ldots N_n) = \frac{\sum_{i=1}^{n} \omega_i (1+T(N_i)) \cdot N_i}{\sum_{i=1}^{n} \omega_i (1+T(N_i))} = \frac{\sum_{i=1}^{n} \omega_i (1+T(N_i)) \cdot a_i}{\sum_{i=1}^{n} \omega_i (1+T(N_i))} + \frac{\sum_{i=1}^{n} \omega_i (1+T(N_i)) \cdot b_i}{\sum_{i=1}^{n} \omega_i (1+T(N_i))} I \quad (34)$$

where $T(N_i) = \sum_{\substack{j=1 \\ i \neq j}}^{n} \sup(N_i, N_j)$, $\sup(N_i, N_j)$ is the degree to which N_j supports N_i . Particularly, when $\omega = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} \right)$, the NNWPA operator will reduce to neutrosophic number power averaging (NNPA) operator:

$$NNWPA(N_1, \ N_2,\ldots N_n) = \frac{\sum_{i=1}^{n} \frac{1}{n} (1+T(N_i)) \cdot N_i}{\sum_{i=1}^{n} \frac{1}{n} (1+T(N_i))} = \frac{\sum_{i=1}^{n} (1+T(N_i)) \cdot a_i}{\sum_{i=1}^{n} (1+T(N_i))} + \frac{\sum_{i=1}^{n} (1+T(N_i)) \cdot b_i}{\sum_{i=1}^{n} (1+T(N_i))} I$$

Obviously the result obtained by Eq. (33) is still a NN.

The Eq.(34) can be proved by Mathematical induction on as follows:

Proof.

(i) Obviously, when $n = 1$, the Eq. (34) is right.

(ii) Given that when $n = k$, the Eq.(34) is right, i.e.,

$$NNWPA(N_1, \ N_2,\ldots N_n) = \frac{\sum_{i=1}^{n} \omega_i (1+T(N_i)) \cdot N_i}{\sum_{i=1}^{n} \omega_i (1+T(N_i))} = \frac{\sum_{i=1}^{n} \omega_i (1+T(N_i)) \cdot a_i}{\sum_{i=1}^{n} \omega_i (1+T(N_i))} + \frac{\sum_{i=1}^{n} \omega_i (1+T(N_i)) \cdot b_i}{\sum_{i=1}^{n} \omega_i (1+T(N_i))}$$

Then when $n = k+1$, we have

$$NNWPA(N_1, N_2, \ldots, N_{k+1}) = NNWPA(N_1, N_2, \ldots, N) + \omega_{k+1} N_{k+1}$$

$$NNWPA(N_1, N_2, \ldots N_{k+1}) = \frac{\sum_{i=1}^{k} \omega_i (1+T(N_i)) \cdot a_i}{\sum_{i=1}^{k} \omega_i (1+T(N_i))} + \frac{\sum_{i=1}^{k} \omega_i (1+T(N_i)) \cdot b_i}{\sum_{i=1}^{k} \omega_i (1+T(N_i))} I$$

$$+\frac{\omega_{k+1}\left(1+T(N_{k+1})\right)\cdot a_{k+1}}{\sum_{i=1}^{k+1}\omega_i\left(1+T(N_i)\right)}+\frac{\omega_{k+1}\left(1+T(N_{k+1})\right)\cdot b_{i+1}I}{\sum_{i=1}^{k+1}\omega_i\left(1+T(N_i)\right)}=\frac{\sum_{i=1}^{k+1}\omega_i\left(1+T(N_i)\right)\cdot N_i}{\sum_{i=1}^{k+1}\omega_i\left(1+T(N_i)\right)}$$

So, when $n=k+1$, the Eq.(34) is right too.

According to (i) and (ii), we can get that the Eq.(34) is right for all n.

Theorem 9. If $\mathrm{Sup}\left(\tilde{a}_k,\tilde{a}_j\right)=c$, $c\in[0,1], k\neq j$,then the weighted power averaging operator of NNs will be reduced to the weighted arithmetic averaging operator of NNs(NNWAA) as follows:

$$NNWPA\left(N_1,N_2,\ldots,N_n\right)=\sum_{i=1}^{n}\omega_i N_i$$

Similarly, we can prove the NNWPA operator has the following characteristics.

Theorem 10. (Idempotency).

Let all $N_i=a+bI$, then

$$NNWPA(N_1,N_2,\ldots N_n)=N_i=a+bI$$

Proof:

Since all $N_i = a_i+b_iI=a+bI$, then we have

$$NNWPA(N_1,N_2,\ldots N_n)=\frac{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)\cdot N_i}{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)}=\frac{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)\cdot a}{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)}+\frac{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)\cdot b}{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)}I=a+bI$$

which completes the proof of theorem 10.

Theorem 11. (Monotonicity).

Let $N_i=a_i+b_iI$ and $N_i^*=a_i^*+b_i^*I$ be two sets of NNs which satisfies $a_i\le a_i^*$, $b_i^*\le b_i$,for all i then

$$NNWPA(N_1,N_2,\ldots N_n)\le NNWPA(N_1^*,N_2^*,\ldots N_n^*)$$

Proof.

Since $a_i\le a_i^*$, $b_i^*\le b_i$, for all i we can get

$$\frac{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)a_i}{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)}\le\frac{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)a_i^*}{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)}$$

$$\frac{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)b_i^*}{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)}\le\frac{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)b_i}{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)}$$

So, we can get

$$NNWPA(N_1, N_2, \ldots N_n) \le NNWPA(N_1^*, N_2^*, \ldots N_n^*).$$

which completes the proof of theorem 11.

Theorem 12. (Boundedness).

Let $N_i = a_i + b_i I$ $(i = 1,2,..., n)$ be a set of NNs. If

$$N_{\max} = \max(N_1, N_2, \ldots N_n) = a_{\max} + b_{\max}$$
$$N_{\min} = \min(N_1, N_2, ..., N_n) = a_{\min} + b_{\max} I,$$

Then

$$N_{\min} \le NNWPA(N_1, N_2, \ldots N_n) \le N_{\max}$$

Proof.

Since for all i, $a_{\min} \le a_i \le a_{\max}$, $b_{\max} \le b_i \le b_{\min}$,

we can get

$$\frac{\sum_{i=1}^{n} \omega_i (1 + T(N_i)) a_{\min}}{\sum_{i=1}^{n} \omega_i (1 + T(N_i))} \le \frac{\sum_{i=1}^{n} \omega_i (1 + T(N_i)) a_i}{\sum_{i=1}^{n} \omega_i (1 + T(N_i))} \le \frac{\sum_{i=1}^{n} \omega_i (1 + T(N_i)) a_{\max}}{\sum_{i=1}^{n} \omega_i (1 + T(N_i))},$$

$$\frac{\sum_{i=1}^{n} \omega_i (1 + T(N_i)) b_{\max} I}{\sum_{i=1}^{n} \omega_i (1 + T(N_i))} \le \frac{\sum_{i=1}^{n} \omega_i (1 + T(N_i)) b_i I}{\sum_{i=1}^{n} \omega_i (1 + T(N_i))} \le \frac{\sum_{i=1}^{n} \omega_i (1 + T(N_i)) b_{\min} I}{\sum_{i=1}^{n} \omega_i (1 + T(N_i))}$$

So, we can get

$$NNWPA(N_{\min}, N_{\min}, \ldots N_{\min}) \le NNWPA(N_1, N_2, \ldots N_n) \le NNWPA(N_{\max}, N_{\max}, \ldots N_{\max})$$

According to theorem 3

$$NNWPA(N_{\min}, N_{\min}, \ldots N_{\min}) = N_{\min}$$

$$NNWPA(N_{\max}, N_{\max}, \ldots N_{\max}) = N_{\max}$$

So, we can get

$$N_{\min} \le NNWPA(N_1, N_2, \ldots N_n) \le N_{\max},$$

which complete the proof of the theorem 12.

3.2 The Neutrosophic Number Weighted Geometric Power Averaging Operator

Definition 8[38]**.** Let $N_i = a_i + b_i I, (i = 1,2, \ldots n)$ be a set of NNs, and NNGPA:NNS^n→NNS. The neutrosophic number geometric power averaging operator is defined as:

$$NNGPA(N_1, N_2, \ldots N_n) = \prod_{i=1}^{n} N_i^{\frac{1+T(N_i)}{\sum_{i=1}^{n}(1+T(N_i))}} \tag{35}$$

where $T(N_i) = \sum_{j=1, j \ne k}^{n} Sup(N_i, N_j)$, the weight of $N_i (i = 1,2, \ldots, n)$ is $\frac{1+T(N_i)}{\sum_{i=1}^{n} 1+T(N_i)}$. Obviously, the NNGPA

operator is a nonlinear weighted-geometric aggregation operator.

Similarly, the NNGPA operator has the characteristics, such as idempotency, monotonicity, boundedness and commutativity.

Theorem 13. (Idempotency).

Let $N_i = a_i + b_i I, (i = 1,2,\ldots n)$ be a set of NNs. If for all i , $N_i = a + bI$, then

$$NNGPA(N_1, N_2, \ldots N_n) = N_i = a + bI .$$

Theorem 14.(Monotonicity).

Let $N_i = a_i + b_i I$ and $N_i^* = a_i^* + b_i^* I$ be two collections of NNs satisfying $a_i \le a_i^*$, $b_i^* \le b_i$, for all $i, i = 1,2,\ldots, n$, then

$$NNGPA(N_1, N_2, \ldots, N_n) = NNGPA(N_1^{'}, N_2^{'}, \ldots N_n^{'}) .$$

Theorem 15. (Boundedness).

Let $N_i = a_i + b_i I$ $(i = 1,2,..., n)$ be a set of NNs, If $N_{\max} = a_{\max} + b_{\min} I$ and $N_{\min} = a_{\min} + b_{\max} I$, then

$$N_{\min} \le NNGPA(N_1, N_2, \ldots N_n) \le N_{\max}$$

Theorem 16. (Commutativity).

Let $(N'_1, N'_2, ..., N'_n)$ be any permutation of $(N_1, N_2, ..., N_n)$, then

$$NNGPA(N_1, N_2, \ldots N_n) = NNGPA(N_1^{'}, N_2^{'}, \ldots, N_n^{'})$$

Definition 9. Let $N_i = a_i + b_i I, (i = 1,2,\ldots n)$ be a set of NNs, and NNGPA:NNS^n→NNS. We define NNWGPA(neutrosophic number weighted geometric power operator) as follows:

$$NNWGPA(N_1, N_2, \ldots, N_n) = \prod_{i=1}^{n} N_i^{\frac{\omega_i(1+T(N_i))}{\sum_{i=1}^{n}\omega_i(1+T(N_i))}} \tag{36}$$

Where $\omega = (\omega_1, \omega_2 \ldots \omega_n)$ is the weighting vector of the N_i , and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weight vector of $N_i (i = 1,2,\ldots n)$ which satisfies $\omega_i \in [0,1]$, $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$. Specially, when $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, the NNWGPA operator will reduce to neutrosophic number geometric power averaging (NNGPA) operator.

Theorem 17. Let $N_i = a_i + b_i I, (i = 1,2,\ldots n)$ be a set of NNs, and Then the result obtained using Eq. (36) is still a NN and

$$NNWGPA(N_1, N_2, \ldots, N_n) = \prod_{i=1}^{n} a_i^{\frac{\omega_i(1+T(N_i))}{\sum_{i=1}^{n}\omega_i(1+T(N_i))}} + \left(\prod_{i=1}^{n} (a_i + b_i)^{\frac{\omega_i(1+T(N_i))}{\sum_{i=1}^{n}\omega_i(1+T(N_i))}} - \prod_{i=1}^{n} a_i^{\frac{\omega_i(1+T(N_i))}{\sum_{i=1}^{n}\omega_i(1+T(N_i))}} \right) \cdot I \tag{37}$$

The proof process is similar to theorem 2, so we can omit it here.

Let

$$\frac{\omega_i\left(1+T(N_i)\right)}{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)}=u_i \tag{38}$$

then the equation turns into:

$$NNWGPA(N_1,N_2,\ldots,N_n)=\prod_{i=1}^{n}a_i^{u_i}+\left(\prod_{i=1}^{n}(a_i+b_i)^{u_i}-\prod_{i=1}^{n}a_i^{u_i}\right)\cdot I \tag{39}$$

Similarly, the NNWGPA operator has the characteristics, such as idempotency, monotonicity, and boundedness.

3.3 The generalized neutrosophic number weighted power averaging operator

Definition 10. Let $N_i=a_i+b_iI\ \ (i=1,2,\ldots,n)$ be a set of NNs, and GNNPA:NNS^n→NNS, If

$$GNNPA(N_1,N_2,\ldots N_n)=\left(\sum_{i=1}^{n}\frac{1+T(N_i)}{\sum_{i}^{n}\left(1+T(N_i)\right)}N_i^{\lambda}\right)^{1/\lambda} \tag{40}$$

where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ is the weight vector of $N_i\,(i=1,2,\ldots,n)$ satisfying $\omega_i\in[0,1]\,(i=1,2,\ldots,n)$

$\sum_{i=1}^{n}\omega_i=1$, and $\lambda\in(0,+\infty)$. Then GNNPA is called generalized neutrosophic number power operator.

Similarly, the GNNPA operator has the commutativity, idempotency and boundedness.

Theorem 18. (Commutativity).

Let $(N'_1,N'_2,\ldots,N'_n)$ be any permutation of $(N_1,N_2,\ldots,N_n)$, then

$$GNNPA(N_1,N_2,\ldots N_n)=GNNPA(N'_1,N'_2,\ldots,N'_n)$$

Theorem 19. (Idempotency).

Let $N_i=a_i+b_iI,(i=1,2,\ldots n)$ be a set of NNs. If for all i, $N_i=a+bI$, then

$$GNNPA(N_1,N_2,\ldots N_n)=a+bI\,.$$

Theorem 20. (Boundedness).

Let $N_i=a_i+b_iI\ \ (i=1,2,\ldots,n)$ be a set of NNs, If $N_{\max}=a_{\max}+b_{\min}I$ and $N_{\min}=a_{\min}+b_{\max}I$, then

$$N_{\min}\le GNNPA(N_1,N_2,\ldots N_n)\le N_{\max}$$

Definition 11. Let $N_i=a_i+b_iI\ \ (i=1,2,\ldots,n)$ be a set of NNs, and GNNWPA:NNS^n→NS, If

$$GNNWPA(N_1,N_2,\ldots N_n)=\left(\sum_{i=1}^{n}u_iN_i^{\lambda}\right)^{1/\lambda} \tag{41}$$

where $u_i=\frac{\omega_i\left(1+T(N_i)\right)}{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)}$, $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ is the weight vector of $N_i\,(i=1,2,\ldots,n)$

satisfying $\omega_i\in[0,1](i=1,2,\ldots,n)$, $\sum_{i=1}^{n}\omega_i=1$ and $\lambda\in(0,+\infty)$. Then GNNWPA is called generalized

neutrosophic number weighted power operator.

Theorem 21. Let $N_i = a_i + b_i I \ (i = 1,2,\dots, n)$ be a set of NNs, and $\lambda \in (0,+\infty)$. Then the result obtained by Eq. (41) is still an NN and

$$GNNWPA(N_1, N_2,\dots, N_n) = (\sum_{i=1}^{n} u_i a_i^{\lambda})^{1/\lambda} + \left(\left(\sum_{i=1}^{n} u_i (a_i + b_i)^{\lambda} \right)^{1/\lambda} - (\sum_{i=1}^{n} u_i a_i^{\lambda})^{1/\lambda} \right) I \tag{42}$$

The proof is similar to the theorem 2, it is omitted here.

Obviously, there are some properties for the GNNWPA operator as follows.

(1) When $\lambda \to 0$,

$$GNNWPA(N_1, N_2,\dots, N_n) = (\sum_{i=1}^{n} u_i N_i^{\lambda})^{1/\lambda} = \prod_{i=1}^{n} a_i^{u_i} + \left(\prod_{i=1}^{n} (a_i + b_i)^{u_i} - \prod_{i=1}^{n} a_i^{u_i} \right) I = \prod_{i=1}^{n} N_i^{u_i}$$

So, the GNNWPA operator is reduced to the NNWGPA operator.

(2) When $\lambda = 1$,

$$GNNWPA(N_1, N_2,\dots, N_n) = (\sum_{i=1}^{n} u_i N_i^{\lambda})^{1/\lambda} = \sum_{i=1}^{n} u_i a_i + \sum_{i=1}^{n} u_i b_i I = \sum_{i=1}^{n} u_i N_i$$

So, the GNNWPA operator is reduced to the NNWPA operator.

In the following, we give some properties of the GNNWPA operator.

Theorem 22. (Idempotency).

Let $N_i = a_i + b_i I, (i = 1,2,\dots n)$ be a set of NNs. If for all i, $N_i = a + bI$, then

$$GNNWPA(N_1, N_2,\dots N_n) = a + bI .$$

Theorem 23. (Boundedness).

Let $N_i = a_i + b_i I \ (i = 1,2,\dots, n)$ be a set of NNs, If $N_{\max} = a_{\max} + b_{\min} I$ and $N_{\min} = a_{\min} + b_{\max} I$, then

$$N_{\min} \le GNNWPA(N_1, N_2,\dots N_n) \le N_{\max}$$

4. Multiple attribute group decision-making method based on GNNWPA operator

In this section, we will provide an illustrative example by applying the power operator under neutrosophic numbers. Suppose that $A = \{A_1, A_2,\dots, A_m\}$ is a set of alternatives, $C = \{C_1, C_2,\dots, C_n\}$ is a set of attributes, and $D = \{D_1, D_2,\dots, D_s\}$ is the set of decision makers.

We use neutrosophic number $N_{ij}^k = a_{ij}^k + b_{ij}^k I$, $a_{ij}^k, b_{ij}^k \in R$ $(k = 1,2,\dots, s;\ j = 1,2,\dots, n;\ i = 1,2,\dots, m)$ to express evaluation value came from the kth $k = (1,2,\dots, s)$ decision maker for the alternative A_i $(i = 1,2,\dots, m)$ under the attribute C_j $(j = 1,2,\dots,n)$ by using a scale from 1 (less fit) to 10 (more fit) with indeterminacy I. Thus, we can get the kth neutrosophic number decision matrix N^k:

$$N^k=\begin{bmatrix} N_{11}^k & N_{12}^k \cdots N_{1n}^k \\ N_{21}^k & N_{22}^k \cdots N_{2n}^k \\ \vdots & \vdots \quad\quad \vdots \\ N_{m1}^k & N_{m2}^k \cdots N_{mn}^k \end{bmatrix}$$

Because each attribute $C_j\,(j=1,2,\ldots,n)$ has different importance, the attribute weight vector is $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ with $\omega_i\in[0,1](i=1,2,\ldots,n)$ and $\sum_{i=1}^{n}\omega_i=1$. Similarly, the weights of decision makers represent the different importance of each decision maker D_k $(k=1,2,..,s)$, and the weighting vector of decision makers is $w=(w_1,w_2,\ldots,w_n)$ with $w_j\geq 0$, $\sum_{j=1}^{n}w_j=1$.

The method of the decision making method involves the following steps:

Step 1: Normalize decision matrix with equation (15), we have

$$N_i=\frac{a_i}{\max(a_i)}+\frac{b_i}{\max(b_i)}I$$

Step 2: Calculate $d(N_{ij}^k,N_{if}^k)$, $T(N_{ij}^k),U(N_{ij}^k)$ with equation (16) (24) (38), we have

$$d(N_i,N_j)=\frac{1}{2}\sqrt{\frac{[(a_j-a_i)+(b_j-b_i)\beta^-]^2+[(a_j-a_i)+(b_j-b_i)\beta^+]^2}{2}}$$

$$T\left(N_i\right)=\sum_{\substack{j=1\\ j\neq i}}^{n}\sup\left(N_i,N_j\right)$$

$$u_i=\frac{\omega_i\left(1+T(N_i)\right)}{\sum_{i=1}^{n}\omega_i\left(1+T(N_i)\right)}$$

Step 3: Utilize the GNNWPA operator, we have

$$N_{ij}^k=a_{ij}^k+b_{ij}^kI=GNNWPA(N_{i1}^k,N_{i2}^k,\ldots,N_{im}^k)$$

to obtain the comprehensive values of each decision maker: N_i^k $(i=1,2,...,m;k=1,2,...,s)$.

Step 4: Utilized the GNNWPA operator, we have

$$N_i=a_i+b_iI=GNNWPA(N_i^1,N_i^2,\ldots,N_i^s)$$

to obtain the collective overall values of each alternatives: N_i $(i=1,2,...,m)$.

Step 5: Calculate the possibility degree $P_{ij}=P(N_i\geq N_j)$, we have

$$P_{ij}=P(N_i\geq N_j)=\max\left\{1-\max\left(\frac{(a_j+b_j\beta^+)-(a_i+b_i\beta^-)}{(a_i+b_i\beta^+)-(a_i+b_i\beta^-)+(a_j+b_j\beta^+)-(a_j+b_j\beta^-)},0\right),0\right\}$$

Step 6: Calculate the values of q_i $(i=1,2,...,m)$ for ranking the orders, we have

$$q_i=\frac{\left(\sum_{j=1}^{n}P_{ij}+\frac{n}{2}-1\right)}{n(n-1)}$$

Step 7: Rank the values of q_i $(i=1,2,...,m)$ in descending order according, and then the best alternative is obtained.

5. A numerical example

We use the generalized neutrosophic number weighted power averaging operator to deal with multiple attribute group decision making problems. An investment company wants to choose a best investment project from four possible alternatives: (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company. There are three attributes that the investment company wants to take into consideration: (1) C_1 is the risk factor; (2) C_2 is the growth factor; (3) C_3 is the environmental factor. The weighting vector of the attributes is $\omega=(0.35,0.25,0.4)$. The company invites three experts $\{D_1,D_2,D_3\}$ to evaluate the four alternatives. The expert weight vector is $w=(0.37,0.33,0.3)$. The $kth(k=1,2,3)$ expert evaluates these four potential alternatives in terms of these three attributes by the form of neutrosophic number $N_{ij}^k=a_{ij}^k+b_{ij}^k$ for $a_{ij}^k,b_{ij}^k\in R$, $k=(1,2,3)$ $i=(1,2,3,4)$ $j=(1,2,3)$ and the evaluation values are shown in tables 1-3.

Then we can make the best alternative for this investment.

Table 1 The evaluation values of four alternatives with respect to the three attributes by the expert D_1

	C1	C2	C3
A1	4+I	5	3+I
A2	6	6	5
A3	3	5+I	6
A4	7	6	4+I

Table 2 The evaluation values of four alternatives with respect to the three attributes by the expert D_2

	C1	C2	C3
A1	5	4	4
A2	5+I	6	6
A3	4	5	5+I
A4	6+I	6	5

Table 3 The evaluation values of four alternatives with respect to the three attributes by the expert D_3

	C1	C2	C3
A1	4	5+I	4
A2	6	7	5+I
A3	4+I	5	6
A4	8	6	4+I

5.1 The evaluation steps of the new MAGDM method based on GNNWPA operator

(1) Normalize the decision matrix by equation (15), we can get the normalized decision matrix shown as follows (Tables 4-6).

Table 4 The evaluation values of four alternatives with respect to the three attributes by experts D_1 .

D1	C1	C2	C3
A1	0.8+I	1	0.6+I
A2	1	1	0.83333
A3	0.5	0.83333+I	1
A4	1	0.8571	0.5714+I

Table5 The evaluation values of four alternatives with respect to the three attributes by experts D_2 .

D2	C1	C2	C3
A1	1	0.8	0.8
A2	0.83333+I	1	1
A3	0.8	1	1+I
A4	1+I	1	0.83333

Table 6 The evaluation values of four alternatives with respect to the three attributes by experts D_3 .

D3	C1	C2	C3
A1	0.8	1+I	0.8
A2	0.8571	1	0.7143
A3	0.6667+I	0.83333	1
A4	1	0.75	0.5+I

(2) Calculate $d(N_{ij}^k, N_{if}^k)$, $T(N_{ij}^k)$, $U(N_{ij}^k)$ (24) and (38)

(i) Calculate $d(N_{ij}^k, N_{if}^k)$ by equation (16), we have the results shown in tables 7-9.

Table 7 Results from calculating $d(N_{ij}^1, N_{if}^1)$

i	$d(N_{i1}^1, N_{i2}^1)$	$d(N_{i1}^1, N_{i3}^1)$	$d(N_{i2}^1, N_{i3}^1)$
$i=1$	0.0851	0.1	0.1851
$i=2$	0	0.08333	0.83333
$i=3$	0.1817	0.25	0.0685
$i=4$	0.0714	0.1993	0.1280

Table 8 Results from calculating $d(N_{ij}^2, N_{if}^2)$

i	$d(N_{i1}^2, N_{i2}^2)$	$d(N_{i1}^2, N_{i3}^2)$	$d(N_{i2}^2, N_{i3}^2)$
$i=1$	0.1	0.1	0
$i=2$	0.0685	0.0685	0
$i=3$	0.1	0.1151	0.0158
$i=4$	0.0158	0.0985	0.08333

Table 9 Results from calculating $d(N_{ij}^3, N_{if}^3)$

i	$d(N_{i1}^3, N_{i2}^3)$	$d(N_{i1}^3, N_{i3}^3)$	$d(N_{i2}^3, N_{i3}^3)$
$i=1$	0.1151	0	0.1151
$i=2$	0.0714	0.0567	0.1280
$i=3$	0.0685	0.1517	0.08333
$i=4$	0.125	0.2351	0.1101

(ii) Calculate $T(N_{ij}^k)$, $U(N_{ij}^k)$ by equations (24) and (38), we have

$$T=\begin{bmatrix}1.8149 & 1.7298 & 1.7149\\ 1.9167 & 1.9167 & 1.8333\\ 1.5683 & 1.7497 & 1.6815\\ 1.7292 & 1.8006 & 1.6727\\ 1.8000 & 1.9000 & 1.9000\\ 1.8630 & 1.9315 & 1.9315\\ 1.7849 & 1.8842 & 1.8691\\ 1.8857 & 1.9009 & 1.8182\\ 1.8849 & 1.7698 & 1.8849\\ 1.8719 & 1.8006 & 1.8154\\ 1.7797 & 1.8482 & 1.7649\\ 1.6399 & 1.7649 & 1.6548\end{bmatrix} \qquad U=\begin{bmatrix}0.3778 & 0.3268 & 0.2954\\ 0.3732 & 0.3329 & 0.2939\\ 0.3570 & 0.3409 & 0.3022\\ 0.3691 & 0.3378 & 0.2931\\ 0.3619 & 0.3343 & 0.3039\\ 0.3645 & 0.3329 & 0.3026\\ 0.3624 & 0.3348 & 0.3028\\ 0.3720 & 0.3335 & 0.2945\\ 0.3749 & 0.3211 & 0.3040\\ 0.3753 & 0.3264 & 0.2983\\ 0.3676 & 0.3359 & 0.2965\\ 0.3637 & 0.3397 & 0.2966\end{bmatrix}$$

(3) Calculate the comprehensive values N_i^k $(i=1,2,3,4; k=1,2,3)$ of each expert D_k by the equation (42) (suppose $\lambda=1$), we have:

$N_1^1=0.8063+0.6732I$ $\quad N_2^1=0.9510$ $\quad N_3^1=0.7647+0.3409I$ $\quad N_4^1=0.8261+0.2931I$

$N_1^2=0.8724$ $\quad N_2^2=0.9392+0.3645I$ $\quad N_3^2=0.9275+0.3028I$ $\quad N_4^2=9509+0.3720I$

$N_1^3=0.8642+0.3211I$ $\quad N_2^3=0.8612+0.2983I$ $\quad N_3^3=0.8215+0.3676I$ $\quad N_4^3=0.7668+0.2966I$

(4) Calculate the overall values, we can get:

$N_1=0.8457+0.3453I$ $\quad N_2=0.9198+0.2108I$ $\quad N_3=0.8354+0.3364I$ $\quad N_4=0.8488+0.3197I$

(5) Calculate the possibility degree $P_{ij}=P(N_i \geq N_j)$ using equation (21) (suppose $I \in [0.02, 0.04]$), we can get.

$$P=\begin{bmatrix}0.5000 & 0.0000 & 1.0000 & 0.3246\\ 1.0000 & 0.5000 & 1.0000 & 1.0000\\ 0.6754 & 0.0000 & 1.0000 & 0.5000\\ 0.4135 & 0.0000 & 0.0000 & 0.5000\end{bmatrix}$$

(6) Calculate the values of q_i $(i=1,2,..., m)$ using equation (22), we can get.

$$q_1 = 0.2354 \quad q_2 = 0.3750 \quad q_3 = 0.1250 \quad q_4 = 0.2646$$

(7) Rank the four alternatives.

Since $q_2 \succ q_4 \succ q_1 \succ q_3$, the ranking order of the four alternatives is $A_2 \succ A_4 \succ A_1 \succ A_3$. So the best choice is A_2.

5.2 The influence of the parameter λ and the indeterminate range for I on the ordering of the alternatives

Different values of parameter λ are used to express different level of the mentality of decision makers, because the bigger λ is, more optimistic decision makers are. In this section, in order to check to which degree different parameter λ influences decision making results, different values of λ are used to analyze the ordering results shown in table 11. (suppose $I \in [0.02, 0.04]$).

Table 11 Ordering of the alternatives by utilizing the different λ in GNNWPA operator

λ	q_i	Ranking
$\lambda = 0.1$	$q_1 = 0.2917$ $q_2 = 0.375$ $q_3 = 0.125$ $q_4 = 0.2083$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\lambda = 0.7$	$q_1 = 0.2543$ $q_2 = 0.375$ $q_3 = 0.125$ $q_4 = 0.2457$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\lambda = 1.0$	$q_1 = 0.2354$ $q_2 = 0.375$ $q_3 = 0.125$ $q_4 = 0.2646$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\lambda = 1.3$	$q_1 = 0.2200$ $q_2 = 0.375$ $q_3 = 0.125$ $q_4 = 0.2800$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\lambda = 1.5$	$q_1 = 0.2030$ $q_2 = 0.375$ $q_3 = 0.1333$ $q_4 = 0.2886$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\lambda = 1.8$	$q_1 = 0.1858$ $q_2 = 0.375$ $q_3 = 0.1476$ $q_4 = 0.2917$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\lambda = 2.0$	$q_1 = 0.1778$ $q_2 = 0.375$ $q_3 = 0.1556$ $q_4 = 0.2917$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\lambda = 2.5$	$q_1 = 0.1617$ $q_2 = 0.375$ $q_3 = 0.1716$ $q_4 = 0.2917$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 3.0$	$q_1 = 0.1502$ $q_2 = 0.375$ $q_3 = 0.1831$ $q_4 = 0.2917$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 10$	$q_1 = 0.1277$ $q_2 = 0.375$ $q_3 = 0.2183$ $q_4 = 0.2790$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 30$	$q_1 = 0.1776$ $q_2 = 0.2987$ $q_3 = 0.2389$ $q_4 = 0.2846$	$A_2 \succ A_4 \succ A_3 \succ A_1$

From Table 11, we can get the different values of λ may lead to different sequence in GNNWPA operator.

(1) When $0 < \lambda < 1$, the order of the alternatives is $A_2 \succ A_4 \succ A_1 \succ A_3$, and the best choice is A_2.

(2) When $1 \le \lambda \le 2$, the order of the alternatives is $A_2 \succ A_1 \succ A_4 \succ A_3$, and the best choice is A_2.

(3) When $2.5 \le \lambda \le 30$, the order of the alternatives is $A_2 \succ A_4 \succ A_3 \succ A_1$, and the best choice is A_2.

Similar to the parameter λ, with the purpose of checking to which degree different parameter I influences decision making results, the different ranges of I are used to calculate the ordering results shown in table 12. (suppose $\lambda = 1$)

Table 12 Ordering of the alternatives by different indeterminate ranges for I in NNGWPA operator

I	q_i	Ranking
$I=0$	/	$A_2 \succ A_4 \succ A_1 \succ A_3$
$I\in[0,0.2]$	$q_1=0.1944$ $q_2=0.3674$ $q_3=0.2444$ $q_4=0.1938$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$I\in[0,0.4]$	$q_1=0.2210$ $q_2=0.3041$ $q_3=0.2488$ $q_4=0.2262$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$I\in[0,0.6]$	$q_1=0.2327$ $q_2=0.2783$ $q_3=0.2497$ $q_4=0.2393$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$I\in[0,0.8]$	$q_1=0.2388$ $q_2=0.2656$ $q_3=0.2498$ $q_4=0.2458$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$I\in[0,1]$	$q_1=0.2427$ $q_2=0.2580$ $q_3=0.2496$ $q_4=0.2497$	$A_2 \succ A_4 \succ A_3 \succ A_1$

From Table 12, we can get the different values of I may lead to different sequence in GNNWPA operator.

(1) When $I=0$, the order of the alternatives is $A_2 \succ A_4 \succ A_1 \succ A_3$, so the best choice is A_2.

(2) When $I\in[0,0.2]$, the order of the alternatives is $A_2 \succ A_3 \succ A_1 \succ A_4$, so the best choice is A_2.

(3) When $I\in[0,0.4]$, $I\in[0,0.8]$, the order of the alternatives is $A_2 \succ A_4 \succ A_1 \succ A_3$, so the best choice is A_2.

(4) When $I\in[0,1]$, the order of the alternatives is $A_2 \succ A_4 \succ A_3 \succ A_1$ and the best alternative is A_2.

In order to demonstrate the effectiveness of the new method in this paper, we compare the ordering results of the new method with the ordering results of the method proposed by [33]. From the table 12 and the table 13, we can find that the two methods produce different ranking results. What's more, the best choice is different too.

Table 13 The ordering results produced by the old method (proposed by Ye[33]).

I	q_i	Ranking
$I=0$	/	$A_2 \succ A_4 \succ A_3 \succ A_1$
$I\in[0,0.2]$	$q_1=0.1250, q_2=0.3368$ $q_3=0.2083, q_4=0.3298$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$I\in[0,0.4]$	$q_1=0.1250, q_2=0.3301$ $q_3=0.2083, q_4=0.3366$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$I\in[0,0.6]$	$q_1=0.1250, q_2=0.3279$ $q_3=0.2083, q_4=0.3388$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$I\in[0,0.8]$	$q_1=0.1250, q_2=0.3267$ $q_3=0.2083, q_4=0.3399$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$I\in[0,1]$	$q_1=0.1250, q_2=0.3261$ $q_3=0.2083, q_4=0.3406$	$A_4 \succ A_2 \succ A_3 \succ A_1$

The method proposed by Ye[33] is based on de-neutrosophication process, it doesn't realize the importance of the rules of powering operation. The new method proposed in this paper is based on the neutrosophic number generalized weighted power averaging operators. Even the value of I is same, when we change the value of λ, the result is different. The example identifies the validity of the multiple attribute group decision making measure, and it provides the more general and flexible

features as I and λ are assigned different values.

In addition, compared with the existing generalized power aggregation operators [29, 30], Zhou et al. [29] extended GPA operator to uncertain environments, and Zhou and Chen [30] extended GPA operator to linguistic environment. Obviously, they didn't process the neutrosophic numbers.

6. Conclusions

In this paper, we firstly use neutrosophic numbers to express uncertain or inaccurate evaluation information. Then we propose generalized neutrosophic number weighted power averaging (GNNWPA) operator as a new method to deal with multiple attribute group decision making problems, which can take the relationship between the decision arguments and the mentality of the decision makers into consideration. Since the decision makers have their interest and the actual need, they can assign the different value λ, which makes the result more flexible and reliable. Finally, we use the possibility degree ranking method to choose the best choice. Afterward, we give a numerical example to reveal the practicability of the new method. Especially, we use the different values of λ and different indeterminate ranges for I to analyze the effectiveness. The significance of the paper is that we combine neutrosophic number with power aggregation operators to cope with multiple attribute group decision making problems. For further research, we will extend GPA operator to refined neutrosophic numbers or interval neutrosophic numbers, and we will also extend other aggregation operators to neutrosophic numbers.

Acknowledgment

This paper is supported by the National Natural Science Foundation of China (Nos. 71471172 and 71271124), the Humanities and Social Sciences Research Project of Ministry of Education of China (No. 13YJC630104), Shandong Provincial Social Science Planning Project (No. 13BGLJ10), the National Soft Science Research Project of China (2014GXQ4D192), the Natural Science Foundation of Shandong Province (No. ZR2014JL046), and Graduate education innovation projects in Shandong Province (SDYY12065). The authors also would like to express appreciations to the anonymous reviewers and Editors for their very helpful comments that improved the paper.

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